To prove Theorem 3 we use equality (4) and direct estimates. The theorem on asymptotic stability can be formulated analogously.

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# ON CERTAIN IMPULSE OBSERVATION LAWS

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We consider the problem of optimizing phase coordinate bounds. We obtain the conditions for the solvability of the problem and establish the form of the optimal observation laws. The paper is closely related to [1, 2]. The problem of optimizing the observation process has been studied from another viewpoint in [3, 4].

**1.** Let a plant's phase coordinate vector $x(t)$ from an $n$-dimensional Euclidean space $R_n$ be the solution of the system of equations

$$x^{\cdot}(t) = A(t)\, x(t) + b(t), \qquad x(0) = x_0 \qquad (0 \leqslant t \leqslant T) \tag{1.1}$$

The vector $y(t)$ accessible to observation is given by the relations

$$dy(t) = h(t)\, H(t)\, x(t)\, dt + \sigma(t)\, d\xi(t), \quad y(0) = 0 \tag{1.2}$$

The elements of the matrices $A(t)$, $H(t)$, $\sigma(t)$ and $b(t)$ are continuous functions. The random variable $x(0)$ has a Gaussian distribution with the covariance matrix

$$D_0 = M(x_0 - Mx_0)(x_0 - Mx_0)', \; D_0 > 0$$

Here the prime is the sign for transposition, $M$ is the mean, the symbol $D_0 > 0$ signifies the positive definiteness of matrix $D_0$. The Wiener process $\xi(t)$ does not depend upon $x(0)$, and the matrix $\sigma(t)\sigma'(t) > 0$, $0 \leqslant t \leqslant T$. Without loss of generality [2] we can take the dimension of vector $y(t)$ equal to $n$. The control of the observation process is effected by choice of the scalar function $h(t)$. We consider the linear combination $q'x(T)$ (the nonzero vector $q \in R_n$) is specified). Let $D(T)$ be the covariance matrix of the conditional distribution of vector $x(T)$ under condition $y(s)$, $0 \leqslant s \leqslant T$.

Problem 1. Determine the function $\gamma(t) = h^2(t)$ (the optimal observation law) which minimizes the expression

$$q'D(T)\, q \tag{1.3}$$

such that

$$\int_0^T \gamma(t)\,dt \leqslant N \tag{1.4}$$

where the constant $N \geqslant 0$ is known.

Functional (1.3) equals the conditional variance of the quantity $q'x(T)$ of interest to the observer. The integral (1.4) yields the quality of the control of the observation. This integral has a simple mechanical sense. Namely [5], integral (1.4) equals the total number of measurements on the interval $[0, T]$. Therefore, requirement (1.4) is a constraint on the total number of measurements.

Since the density of observations $\gamma(t)$ at an instant $t$ is not bounded, we assume a priori observations of the form

$$\gamma(t) = \sum_{t_i \leqslant t} \mu_i \delta(t - t_i), \qquad 0 \leqslant t_i \leqslant T$$

where the constants $\mu_i > 0$ and $\delta(t)$ is the delta function. The quantity being observed equals [1]

$$y(t_i) = \sqrt{\mu_i} H(t_i) x(t_i) + \sigma(t_i) \zeta(t_i)$$

where $\zeta(t_i)$ is a sequence of mutually-independent equally-distributed Gaussian variables with zero mean and unit covariance matrix.

**2.** By $z(t, s)$ we denote the fundamental matrix of the homogeneous system (1.1) for $b(t) = 0$ and we set

$$Q = \int_0^T z'(s, T) V(s) z(s, T)\,ds$$

where

$$V(s) = H'(s)\,\sigma(s)\sigma'(s)^{-1} H(s)$$

The paper main result is the following.

**Theorem.** Assume that the coefficients of Eqs. (1.1), (1.2) satisfy the requirements of Sect. 1 and that the matrix $Q > 0$. Then the optimal observation law $\gamma(t)$ solving Problem 1 has the form

$$\gamma(t) = \sum_{i=1}^{m} \mu_i \delta(t - t_i), \qquad 0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_m \leqslant T \tag{2.1}$$

where the constants $\mu_i \geqslant 0$ and $\mu_1 + \mu_2 + \ldots + \mu_m = N$, while the integer $m \leqslant {}^1/_2\, n(n+1)$.

The proof of the theorem consists of four stages.

**1°.** By $r(t)$ we denote the matrix satisfying the equation

$$dr(t) = [-r(t)A(t) - A'(t)\,r(t)]\,dt + V(t)\,du(t)$$
$$r(0) = D_0^{-1} \quad (0 \leqslant t \leqslant T) \tag{2.2}$$

in which the scalar control $u(t)$ is chosen from a set $U$ of nondecreasing functions of bounded variation on the interval $[0, T]$, equal to zero for $t = 0$. We recall that equations of form (2.2) are to be understood in the sense of the corresponding integral identity, while the integral

$$\int_0^t V(s)\,du(s)$$

is a Lebesgue-Stieltjes integral. By virtue of the condition $r(0) > 0$, of the definition of set $U$ and of Eq. (2.1), the matrix $r(t, u) > 0$, $0 \leqslant t \leqslant T$, for any function $u \in U$.

Sometimes the solution of Eq. (2.2) will be denoted by the symbol $r(t, u)$ to emphasize its dependence on the control $u(t)$. We set

$$\|u\| = \int_0^T |du(t)|$$

The aim of the first stage of the proof is to establish the existence of a control $u(t) \in U$ which solves an auxiliary Problem 2, namely find a function $u(t) \in U$, minimizing the functional $q'r(T, u)^{-1} q$, such that $\|u\| \leqslant N$.

Let $U_1$ be a set of functions from $U$, satisfying the requirement $\|u\| \leqslant N$. We introduce a sequence $u_i(t)$, $i=1, 2, \ldots$ of functions from $U_1$ by means of the relations

$$\lim_{i\to\infty} q'r(T, u_i)^{-1} q = \inf_{U_1} q'r(T, u)^{-1} q = I$$

From the definition of set $U_1$ it follows that all the functions $u_i(t)$, $0 \leqslant t \leqslant T$, as well as their norms, are bounded by the number $N$. From this and from Helly's second theorem it follows that some subsequence of the sequence $u_i(t)$ converges to $u_0(t)$ at each point of the interval $[0, T]$. For brevity let us accept that it is the sequence $u_i(t)$ itself that converges to $u_0(t)$. From this and from Helly's first theorem we conclude that $u_i(t)$ converges to $u_0(t)$ also weakly.

For any points $t_1, t_2, t_2 \geqslant t_1$ of the interval $[0, T]$ we have

$$u_0(t_2) - u_0(t_1) \geqslant u_0(t_2) - u_i(t_2) + u_i(t_1) - u_0(t_1)$$

From this bound and from the pointwise convergence of $u_i(t)$ it follows that the function $u_0(t)$ does not decrease. Hence, $\|u_0\| \leqslant N$. Thus, to prove the optimality of $u_0(t)$ relative to Problem 2 it remains only to verify that $I$ is equal to $q'r(T, u_0)^{-1}q$.

The boundedness, uniform for $0 \leqslant t \leqslant T$ and for all $i = 1, 2, \ldots$ of all elements of matrices $r(t, u_i)$ follows from Eq. (2.2), and from the properties of $u_i(t)$. Therefore, we can find a constant $c > 0$ such that

$$\left| \int_{t_1}^{t_2} (r(t, u_i) A(t))_{l, j} \, dt \right| \leqslant c(t_2 - t_1) \tag{2.3}$$

for any point $t_2 \geqslant t_1$ of the interval $[0, T]$ and for any $l, j = 1, 2, \ldots, n$. Here $Q_{lj}$ denotes the $lj$ element of matrix $Q$. From bounds (2.3) we see (cf [6], p. 82) that the integrals

$$\int_0^t r(s, u_i) A(s) \, ds$$

as functions of $t$ are of bounded variation of the interval $[0, T]$; the variation is uniform in view of the uniform boundedness of the elements of matrices $r(t, u_i)$ and of the theorem's hypotheses. We can establish analogously that the expressions

$$\int_0^t V(s) \, du_i(s), \qquad \int_0^t A'(s) r(s, u_i) \, ds$$

are of bounded variation, uniformly in $i$, on $[0, T]$. From this and from Eq. (2.1) it follows that the elements of matrices $r(t, u_i)$ also are of bounded variation, uniformly in $i = 1, 2, \ldots$, on $[0, T]$.

Hence, using the arguments which were applied above to the sequence $u_i(t)$, we can show (passing, if necessary, to a subsequence) that $r(t, u_i)$ converges pointwise to $r_0(t)$.

Then, from the weak convergence of $u_i(t)$ to $u_0(t)$ and from Lebesgue's theorem on passing to a limit under the integral sign, it follows that

$$\lim_{i\to\infty}\left(\int_0^t [-r(s,u_i)A(s)-A'(s)r(s,u_i)]\,ds+\int_0^t V(s)\,du_i(s)\right)=$$
$$\int_0^t (-r_0(s)A(s)-A'(s)r_0(s))\,ds+\int_0^t V(s)\,du_0(s)$$

From the last equality, from the convergence of $r(t,u_i)$ to $r_0(t)$, and from Eq. (2.2) it follows that $r_0(t)=r(t,u_0)$. By the same token we have established the equality

$$I=q'r(T,\ u_0)^{-1}q \tag{2.4}$$

2°. We now prove that

$$\|u_0\|=N \tag{2.5}$$

Let us assume to the contrary that $\|u_0\|<N$ and show that in this case we can find a function $u_1\in U_1$ for which

$$q'r(T,\ u_1)^{-1}q<q'r(T,\ u_0)^{-1}q \tag{2.6}$$

It is clear that bound (2.6) is impossible since it contradicts equality (2.4), established in stage 1°, and the definition of the number $I$. We assume $\varepsilon$ being a constant,

$$u_1(t)=u_0(t)+\varepsilon t,\qquad \varepsilon=T^{-1}(N-\|u_0\|)>0$$

It is easily verified that $u_1\in U_1$. Further, using the equality

$$q'r(T,\ u_1)^{-1}q=\max_{y\in R_n}[2y'q-y'r(T,\ u_1)] \tag{2.7}$$

we obtain, in view of (2.2) and of the definition of matrix $Q$, that

$$q'r(T,u_1)^{-1}q=\max_{y\in R_n}[2y'q-y'r(T,u_0)-y'\varepsilon Qy]<$$
$$\max_{y\in R_n}[2y'q-y'r(T,u_0)y]=q'r(T,u_0)^{-1}q$$

Equality (2.5) is established.

3°. We fix the matrix $r(T,u_0)$ and we consider an auxiliary Problem 3, namely to find a function $\omega(t)$, $\omega(0)=0$, with minimal norm, such that

$$r(0,\ \omega)=r(0),\qquad r(T,\ \omega)=r(T,\ u_0)$$

We emphasize that in Problem 3 we seek the optimal $\omega(t)$ among all functions of bounded variation on $[0,\ T]$ and not just the monotonic ones, as was the case in Problems 1 and 2.

The aim of the third stage of the proof is to substantiate that $u_0(t)$ solves Problem 3. First of all, it is clear that the function $u_0(t)$ is admissible for Problem 3, since $\|u_0\|=N<\infty$, $r(0,\ u_0)=r(0)$, and at the instant $T$ the solution of (2.2), corresponding to $u_0$, equals $r(T,\ u_0)$. From the existence of the admissible function we can establish, analogously as in stage 1° of the proof, the existence of the optimal $\omega_0(t)$ for Problem 3, and since $\|u_0\|=N$, we have $\|\omega_0\|\leqslant N$.

The validity of the third stage of the proof will be established if we show, firstly, that the solution of Problem 3 is yielded by a nondecreasing function and, secondly, that $u_0(t)$ is optimal for Problem 3 in the class $U_1$. Let us assume that $\omega_0(t)$ is not an increasing function. We then set

$$\omega_0(t)=\omega_{01}(t)-\omega_{02}(t),\ \omega_{01}(0)=\omega_{02}(0)=0\ (0\leqslant t\leqslant T)$$

$$2\omega_{01}(t) = \int_0^t |d\omega_0(t)| + \omega_0(t) \quad 2\omega_{02}(t) = \int_0^t |d\omega_0(t)| - \omega_0(t)$$

and, by virtue of our assumption, the function $\omega_{02}(t)$ has points of growth on $[0, T]$, i.e. $\|\omega_{02}\| > 0$; consequently, $\|\omega_{01}\| < N$. On the basis of (2.7) we have

$$q'r(T, u_0)^{-1} q = \max_{y \in R_n} [2y'q - y'r(T, u_0) y]$$

From this and from (2.2), and because $\omega_{02}(t)$ is nondecreasing, we conclude that

$$I = q'r(T, u_0)^{-1} q =$$

$$\max_{y \in R_n} \left[2y'q - y'z'(0, T) r(0) z(0, T) y - y' \int_0^T z'(s, T) V(s) z(s, T) d\omega_0(s) y\right] \geqslant$$

$$\max_{y \in R_n} \left[- y' \int_0^T z'(s, T) V(s) z(s, T) d\omega_{01}(s) y + 2y'q - y'z'(0, T) r(0) z(0, T) y\right] =$$

$$q'r(T, \omega_{01})^{-1} q$$

Hence, the nondecreasing function $\omega_{01} \in U_1$ solves Problem 2. However, $\|\omega_{01}\| < N$. Therefore, by a verbatim repetition of the arguments in stage 2° of the proof of the theorem (with $u_0$ replaced by $\omega_{01}$), we are convinced of the existence of a function $u_1 \in U_1$, $\|u_1\| = N$, for which

$$q'r(T, u_1)^{-1} q < q'r(T, \omega_{01})^{-1} q \leqslant q'r(T, u_0)^{-1} q$$

The latter is obviously impossible because it contradicts the optimality of $u_0(t)$ in Problem 2. The optimality of $u_0(t)$ in the class $U_1$ is established in like manner.

4°. On the basis of stages 1° and 2° of the proof the solution of Problem 2 is yielded by a nondecreasing function with a norm equal to $N$. On the other hand, by virtue of stage 3°, the solving of Problem 2 is equivalent to the solving of Problem 3, which in the usual fashion reduces to the moment problem in view of the linearity of Eq. (2.2) (see [3, 7]). Therefore, with due regard to [7] and to stages 1° – 3°, the optimal function solving Problem 2 is a nondecreasing piecewise-constant function with a norm equal to $N$ and with a number of jumps not exceeding $^1/_2\, n(n+1)$.

Finally, let us ascertain the relation of Problem 1 to Problem 2. By virtue of [1] the matrix $D^{-1}(t)$ is a solution of the system of equations

$$D^{\cdot -1}(t) = - D^{-1}(t) A(t) - A'(t) D^{-1}(t) + V(t) \gamma(t) \tag{2.8}$$

Any nonnegative function $\gamma(t)$ satisfying requirement (1.4) can be associated with a function $u(t)$ of bounded variation on $[0, T]$

$$u(t) = \int_0^t \gamma(s)\, ds, \qquad u(0) = 0 \tag{2.9}$$

and, in view of (1.4), $\|u\| \leqslant N$. Let us extend the set of functions $\gamma(t)$ up to $U_2 \subset U_1$ by formula (2.9). Then Problem 1 turns into Problem 2. The inverse correspondence holds not for any function $u(t)$ of bounded variation but only for those which do not contain a singular component [8]. In particular, such a correspondence holds for piecewise-constant functions $u(t)$. From this and from the established form of the optimal function in Problem 2 it follows that the optimal function $\gamma(t)$ in Problem 1 can be chosen in accordance with (2.1). The theorem is proved.

3. One of the hypotheses in the theorem proved in Sect. 2 is the requirement $Q > 0$. Let us formulate certain conditions in terms of the coefficients of Eqs. (1.1), (1.2), under whose fulfillment the matrix $Q$ is positive definite.

Lemma. Assume that the matrices $A$, $H$, $\sigma$ are constant and satisfy the requirements of Sect. 1. Then $Q > 0$ if and only if the rank of the matrix

$$R_1 = (H', A'H', \ldots, (A')^{n-1}H')$$

equals the number $n$, namely, the dimension of system (1.1).

Proof. From the results in [2] it follows that the necessary and sufficient condition for the positive definiteness of $Q$ is that the matrix

$$R_2 = (H' (\sigma')^{-1}, A'H' (\sigma')^{-1}, \ldots, (A')^{n-1} H' (\sigma')^{-1})$$

be of full rank. It is also clear that from the condition $\sigma\sigma' > 0$ follows the nonsingularity of matrix $\sigma$. Hence, it is sufficient to show that the ranks of matrices $R_1$ and $R_2$ are equal for any nonsingular $\sigma$.

By $R_3$ we denote a block diagonal matrix of dimension $n^2 \times n^2$ with elements $(\sigma')^{-1}$ on the main diagonal. The rank of matrix $R_3$ equals $n^2$ because $\sigma$ is nonsingular. Moreover,

$$R_2 = R_1 R_3 \tag{3.1}$$

Thus, the equality of the ranks of matrices $R_1$ and $R_2$ follows from formula (3.1) and from Sylvester's inequality ( [9], p. 57). The lemma is proved.

Using [3], analogously to the proof of the lemma, we can obtain certain conditions for the positive definiteness of matrix $Q$ also for the case of variable coefficients $A$, $H$, $\sigma$. For example, let the functions $A$, $H$, $\sigma$ satisfy the requirements in Sect. 1 and let there exist a point $s \in [0, T]$ in some neighborhood of which the derivatives of the matrices $A$, $H$ up to order $n - 1$ are continuous, while at the point $s$ the rank of the matrix

$$(K_1 (s), \ldots, K_n (s))$$

equals the number $n$, where

$$K_1 (s) = H' (s), \quad K_{i+1} (s) = \frac{dK_i (s)}{ds} + A' (s) K_i (s)$$

Then matrix $Q > 0$.

The theorem proved in Sect. 2 reduces the question of an optimal observation law to the problem of minimizing a scalar function of a finite number of variables. For this we should solve Eq. (2.8) with $\gamma (t)$ equal to (2.1). As a result functional (1.3) proves to be a scalar function of the variables $\mu_i$ and $t_i$. Let us illustrate what we have said by examples.

Example 1. Let Eqs. (1.1), (1.2) be scalar with constant coefficients, where $H (t) = \sigma (t) = 1$. Then, according to Theorem 1 the optimal observation law has the form $\gamma (t) = N\delta (t - t_1)$. Substituting this $\gamma (t)$ into (2.8), we obtain that $t_1 = T$ when $a > 0$; $t_1 = 0$ if $a < 0$ ; while for $a = 0$ the value of functional (1.3) does not depend upon the actual instant the observations are made.

Example 2. Let Eqs. (1.1), describing the free motion of a material point on a straight line, have the form

$$x_1^{\cdot} (t) = x_2 (t), \quad x_2^{\cdot} (t) = 0 \tag{3.2}$$

Matrix $D(0)$ is diagonal with diagonal elements $d_1, d_2$ equal to the a priori variances of the coordinate and of the velocity, respectively. We assume that the coordinate is observed, i.e.

$$dy(t) = h(t)x_1(t)\,dt + d\xi(t) \qquad (3.3)$$

and we are required to minimize the variance of the velocity $x_2(T)$ at the end of the observation process.

According to the theorem the optimal observation law $\gamma(t)$ solving Problem 1 (in which $q' = (0, 1)$) for the system (3.2), (3.3) has the form

$$\gamma(t) = \mu_3\delta(t - t_3) + \mu_1\delta(t - t_1) + \mu_2\delta(t - t_2),$$

where $t_i$ are certain points of the interval $[0, T]$ and the nonnegative constants $\mu_i$ are subject to the requirement $\mu_1 + \mu_2 + \mu_3 = 1$. From this and from Eq. (2.8) we obtain

$$D(T) = z(T, 0)\left[(D(0))^{-1} + \int_0^T z'(\tau, 0) V_1 z(\tau, 0)\gamma(\tau)\,d\tau\right]^{-1} z'(T, 0) \qquad (3.4)$$

where the elements $v_{ij}$ of matrix $V_1$ are equal zero except for $v_{11} = 1$. The elements $z_{ij}(t, 0)$ of the matrix $z(t, 0)$ are

$$z_{11}(t, 0) = z_{22}(t, 0) = 1, \; z_{12}(t, 0) = t, \; z_{21}(t, 0) = 0$$

Carrying out the simple calculations we obtain from (3.4) that Problem 1 is reduced to the determination of the numbers $\mu_i$, $t_i$ which maximize the function

$$\Delta = (d_1^{-1} + 1)(d_2^{-1} + \mu_1 t_1^2 + \mu_2 t_2^2 + \mu_3 t_3^2) - (\mu_1 t_1 + \mu_2 t_2 + \mu_3 t_3)^2$$

$$\mu_i \geqslant 0, \; \mu_1 + \mu_2 + \mu_3 = 1, \; 0 \leqslant t_i \leqslant T, \; i = 1, 2, 3 \qquad (3.5)$$

Here $\Delta$ is the determinant of the matrix occurring within the brackets in (3.4).

However, for any fixed $\mu_i$ satisfying constraints (3.5), the function in the variables $t_i$ $(t_i \neq 0)$

$$(d_1^{-1} + 1)(\mu_1 t_1^2 + \mu_2 t_2^2 + \mu_3 t_3^2) - (\mu_1 t_1 + \mu_2 t_2 + \mu_3 t_3)^2 \qquad (3.6)$$

is positive on the basis of Sylvester's criterion for positive definiteness (see [9], p. 151). In other words, the quadratic form (3.6) in the variables $t_i$ is positive definite for any fixed $\mu_i$, i.e. its maximum with respect to $t_i$, $0 \leqslant t_i \leqslant T$, is reached at one of the vertices of the three-dimensional cube $0 \leqslant t_i \leqslant T$, $i = 1, 2, 3$. From this and from the fact that the function

$$(d_1^{-1} + 1)d_2^{-1} + d_1^{-1}\mu T^2, \quad 0 \leqslant \mu \leqslant 1$$

achieves a maximum when $\mu = 1$, it follows that in the original Problem 1 the quantities $t_1 = t_2 = t_3 = T$, i.e. all observations are carried out at the end of the observation process.

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# ON THE POSSIBILITY OF GASDYNAMIC EFFECTS AT THE CRITICAL POINT OF THE PHASE EQUILIBRIUM

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A possibility is indicated of appearance of density excursions in one-dimensional unsteady fluid flows near the critical point of the phase equilibrium, resulting from the singularities in the equation of state.

The present investigations are concerned with the question, whether the classical solutions of the problem and the initial conditions for the one-dimensional unsteady gasdynamic equations can become infinite in the nonisoentropic case. Here we have to consider a system of three quasilinear hyperbolic equations which, as we know [1, 2], usually have unbounded solutions. On the other hand, the system of gasdynamic equations has a number of specific properties. Of those the most important is the presence of a single invariant, i.e. of a function which remains bounded [1]. Another important property consists of the fact that the generalized Riemann invariants satisfy multi-dimensional integral equations of Volterra type, in which the cone of integration is represented by the domain of definition of the hyperbolic equations and the boundedness of the solution follows from the fine properties of the integrability of the kernel. In the terms of the gasdynamic equations the latter lead to restrictions imposed on the equations of state. The properties themselves follow from the boundedness of the variation of entropy along the sonic characteristics and from the weak linearity (tangency) of the entropic characteristics [3].

The conditions which must be imposed on the equations of state in order to secure the boundedness, are expressed by the following inequalities [3]

$$0 < c^\circ \leqslant c_V = T\left(\frac{\partial S}{\partial T}\right)_V, \qquad \left|\left(\frac{\partial^2 p}{\partial \rho \partial S}\right)\left(\frac{\partial p}{\partial \rho}\right)_S^{-1}\right| \leqslant K^\circ \tag{1}$$